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Non-Vanishing and Orthogonal Basis of Symmetry Classes of Tensors

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Abstract. By Cayley's theorem, any finite group G of order n can be regarded as a subgroup of the symmetric group $\mathbb{S}_n$. Let χ be any irreducible complex character of G and let $V_\chi^n(G)$ denote the symmetry classes of tensors associated with G and χ. In this paper assuming the Cayley representation of G, we obtain a formula for the dimension of $V_\chi^n(G)$ and discuss its non-vanishing in general. A necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ is also obtained.

Keywords: symmetry class of tensors, decomposable symmetrized tensor, orthogonal basis, Cayley representation.

1. Introduction

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$. Let $\overset{n}{\otimes}V$ be the nth tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation g in $\mathbb{S}_n$, there corresponds a unique linear operator $P(g) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V$ determined by $P(g)(v_1 \otimes \cdots \otimes v_n) = v_{g^{-1}(1)} \otimes \cdots \otimes v_{g^{-1}(n)}$. Let G be a subgroup of $\mathbb{S}_n$ and $I(G)$ the set of all the irreducible complex characters of G. It follows from the orthogonality relations for characters that

$$\left\{T(G,\chi) : \overset{n}{\otimes}V \to \overset{n}{\otimes}V \mid T(G,\chi) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)P(g),\ \chi \in I(G)\right\}$$

is a set of annihilating idempotents which sum to the identity. The image of $\overset{n}{\otimes}V$ under $T(G,\chi)$ is called the *symmetry class of tensors* associated with G and χ and is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G,\chi)$ is denoted by $v_1 * \cdots * v_n$ and is called a *decomposable tensor*. It is well known that

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|}\sum_{g\in G}\chi(g)m^{c(g)}, \tag{1}$$

where $c(g)$ is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of g (see [7]). Also,

$$\overset{n}{\otimes} V = \bigoplus_{\chi \in I(G)} V_\chi^n(G) \tag{2}$$

is a direct sum.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1 \leq \alpha_i \leq m$ so that α is a mapping from a set of n elements into a set of m elements. Then the group G acts on Γ_m^n by $g \cdot \alpha := (\alpha_{g^{-1}(1)}, \dots, \alpha_{g^{-1}(n)})$, where $g \in G$ is a permutation on n letters and $\alpha \in \Gamma_m^n$ is a mapping from a set of n elements into a set of m elements. Therefore, the action may be written as $g \cdot \alpha = \alpha g^{-1}$ which is a composition of two functions. Let $O(\alpha) = \{g \cdot \alpha \mid g \in G\}$ be the *orbit* with representative α, and also let G_α be the *stabilizer* of α, i.e., $G_\alpha = \{g \in G \mid g \cdot \alpha = \alpha\}$. Let Δ be a system of distinct representatives of the orbits of G acting on Γ_m^n and define

$$\overline{\Delta} = \left\{ \alpha \in \Delta \mid \sum_{g \in G_\alpha} \chi(g) \neq 0 \right\},$$

and let $\hat{\Delta}$ be the union of those equivalence classes represented by elements of $\overline{\Delta}$.

Let $\{e_1, \dots, e_m\}$ be a basis of V. Denote by e_α^* the tensor $e_{\alpha_1} * \cdots * e_{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_m^n$. For $\gamma \in \overline{\Delta}$, $V_\gamma^* = \langle e_{g \cdot \gamma}^* \mid g \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$. It follows that

$$V_\chi^n(G) = \bigoplus_{\gamma \in \overline{\Delta}} V_\gamma^*, \tag{3}$$

is a direct sum. In [4], Freese proved that

$$\dim V_\gamma^* = \frac{\chi(1)}{|G_\gamma|} \sum_{g \in G_\gamma} \chi(g), \tag{4}$$

in particular, if χ is of degree one, then $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$.

A particular case appears when we assume that V is an m-unitary space. In this case, the inner product on V induces an inner product on $\overset{n}{\otimes} V$ whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \cdots * u_n \mid v_1 * \cdots * v_n \rangle = \frac{\chi(1)}{|G|} d_\chi^G(A),$$

where $A = [a_{ij}]_{n \times n} = [\langle u_i | v_j \rangle]_{n \times n}$ and $d_\chi^G(A) = \sum_{g \in G} \chi(g) a_{1g(1)} \cdots a_{ng(n)}$ is the *generalized matrix function.*

With respect to the above inner product, the sums that appeared in (2) and (3) are orthogonal direct sums. Also, if $\{e_1, \dots, e_m\}$ is an orthonormal basis of V, then we obtain

$$\langle e_\alpha^* \mid e_\beta^* \rangle = \begin{cases} \frac{\chi(1)}{|G|} \sum\limits_{\sigma \in G_\beta} \chi(\sigma\tau^{-1}) & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\ 0 & \text{if } O(\alpha) \neq O(\beta). \end{cases}$$

In particular, by taking the norm of e_α^*, with respect to the induced inner product, one can easily obtain the condition $e_\alpha^* \neq 0$ if and only if $\alpha \in \hat{\Delta}$.

If $\alpha = g \cdot \gamma$ and $\beta = g' \cdot \gamma$, then $gg'^{-1} \cdot \beta = \alpha$, and so if we let $\tau = gg'^{-1}$ and use the above formula for $\langle e_\alpha^* \mid e_\beta^* \rangle$, then we obtain

$$\langle e_{g\cdot\gamma}^* \mid e_{g'\cdot\gamma}^* \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\gamma} \chi(g'\sigma g^{-1}). \tag{5}$$

An orthogonal basis of the form $\{e_\alpha^* \mid \alpha \in S\}$, where S is a subset of Γ_m^n, is called an *orthogonal basis of decomposable symmetrized tensors* for $V_\chi^n(G)$. By (3), $V_\chi^n(G)$ has an orthogonal basis of decomposable symmetrized tensors if and only if, for all $\gamma \in \overline{\Delta}$, the orbital subspace V_γ^* has an orthogonal basis of decomposable symmetrized tensors. In particular, if χ is of degree one, since $\dim V_\gamma^* = 1$ for all $\gamma \in \overline{\Delta}$, then V_γ^* has an orthogonal basis of decomposable symmetrized tensors for all $\gamma \in \overline{\Delta}$ which implies that $V_\chi^n(G)$ has such a basis. Non-vanishing of $V_\chi^n(G)$ were studied by several authors and they found a formula for $\dim V_\chi^n(G)$ in a more closed form than (1). Also, the existence of an orthogonal basis of decomposable symmetrized tensors for these vector spaces was considered (see, for example, [1, 14]). In [8] and [10], a formula for $\dim V_\chi^n(G)$ is also given when G is equal to the whole group $\$_n$, and in [13], a formula for calculating $\dim V_\chi^n(G)$ is given in the case that $G = \langle \pi_1 \rangle \cdots \langle \pi_p \rangle$ and in [2] for $G = \langle \pi_1 \ldots \pi_p \rangle$ is given, where π_i's, $1 \le i \le p$, are disjoint cycles in $\$_n$.

Also, in [5] a necessary and sufficient condition for the existence of orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ is given, when G is a cyclic or a dihedral group and in [3] when G is a dicyclic group.

In this paper, we let G be a subgroup of $\$_n$ by acting faithfully on a set of n elements. In this case, the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in I(G)$ and we will discuss the non-vanishing property of these vector spaces. As a special case, that is, when G is a group of order n and acts on G by right multiplication, we can prove that, for all $\chi \in I(G)$, $V_\chi^n(G) \neq 0$, and we will find a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for these vector spaces.

2. Main Results

Let V be an m-dimensional vector space over the complex field $\mathbb{C}$ and let G be a finite group and Ω a set of n elements. Suppose G acts faithfully on Ω, so we can assume that G is a subgroup of $\$_n$, i.e., $G = \{g \mid g \in G\} = \{\sigma_g \mid g \in G\}$, where $\sigma_g : \Omega \to \Omega$ defined by $\sigma_g(\omega) = g \cdot \omega$ for all $\omega \in \Omega$, is a permutation on n letters. Therefore, the vector space $V_\chi^n(G)$ is meaningful for all $\chi \in I(G)$. G as a subgroup of $\$_n$ acts on n letters and we denote by θ its permutation character. For $g \in G$, the value $\theta(g)$ is the number of letters fixed by g, i.e., the number of cycles of length one in the cycle structure of g. In the following lemma we give a formulation of (1) in terms of θ.

Lemma 1. *Let G be a finite group and Ω a set of n elements. Assume that G acts faithfully on Ω and let V be an m-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $\chi \in I(G)$, we have*

$$\dim V_\chi^n(G) = \frac{\chi(1)}{|G|} \sum_{g \in G} \chi(g) m^{(\theta\downarrow_{\langle g \rangle}, 1_{\langle g \rangle})_{\langle g \rangle}}.$$

Proof. Suppose $g = \sigma_g \in G$. Then by definition, $c(g)$ is the number of cycles in the cycle structure of g including cycles of length one. But it is easy to see that this number is equal to the number of orbits of the cyclic group $\langle g\rangle$ acting on Ω. By Burnside's Lemma (see [9, p. 59]) the number of orbits of $\langle g\rangle$ acting on Ω is $1/|\langle g\rangle|\Sigma_{\sigma\in\langle g\rangle}\theta(\sigma)$ and so $c(g) = (\theta\downarrow_{\langle g\rangle}, 1_{\langle g\rangle})_{\langle g\rangle}$, hence the theorem follows by (1). □

As an application of the above lemma, we give here a solution to problem 5.18 in [6] and obtain a number theoretical result.

Proposition 2. *Let G be a finite group. Write* $a(k) = |\{g \in G \mid o(g) = k\}|$. *Then the polynomial* $f(x) = 1/|G|\Sigma_k a(k)x^{|G|/k}$ *takes on integer values whenever* $x \in \mathbb{Z}$.

Proof. First, we claim that $\psi : G \to \mathbb{C}$, defined by $\psi(g) = (-1)^{|G|/o(g)}$, $g \in G$, is a generalized character of G. If $|G|$ is odd, then $\psi(g) = -1$, for all $g \in G$. This leads to $\psi = -1_G$ and so ψ is a generalized character of G. Hence, we suppose $|G|$ is even. Consider $D : G \to GL_n(\mathbb{C})$ the regular representation of G, where $|G| = n$. Then by problem 2.3 of [6], det : $G \to \mathbb{C}$, defined by $\det(g) = \det(D(g))$, $g \in G$, is a linear character of G. But it is easy to see that $\det(D(g)) = (-1)^{(o(g)-1)|G|/o(g)}$, $g \in G$. Since $|G|$ is even, for all $g \in G$, we have $\det(g) = \det(D(g)) = (-1)^{|G|-|G|/o(g)} = (-1)^{|G|/o(g)} = \psi(g)$. Therefore, det $= \psi$ and so in this case, ψ is a linear character of G, and our claim is established.

Now by the notation in Lemma 1, we consider the function $P : g \to P(g)$. It can be easily shown that P is a representation of G affording the character ϕ_m such that $\phi_m(g) = m^{c(g)} = m^{(\theta\downarrow_{\langle g\rangle}, 1_{\langle g\rangle})_{\langle g\rangle}}$, $g \in G$.

Let $\Omega = G$ and consider that G acts on Ω by right multiplication, and put $|G| = n$. In this case, G can be regarded as a subgroup of $\$_n$, whose action on Ω is regular and therefore, $c(1) = n = |G|$ and $c(g) = |G|/o(g)$ for $g \neq 1$. Thereby, for any $\chi \in I(G)$, we have $\dim V_\chi^n(G) = \chi(1)/|G|\Sigma_{g\in G}\chi(g)m^{|G|/o(g)}$. Now, taking χ to be the principal character of G, we obtain $\dim V_\chi^n(G) = 1/|G|\Sigma_{g\in G}m^{|G|/o(g)}$ which is a natural number. For all $x \in \mathbb{Z}$, we have $f(x) = 1/|G|\Sigma_k a(k)x^{|G|/k} = 1/|G|\Sigma_{g\in G}x^{|G|/o(g)}$. Therefore, if $x > 0$, then $f(x) = (\phi_x, 1_G)_G \in \mathbb{N}$ and if $x < 0$, then $f(x) = (\phi_{-x}, \psi)_G \in \mathbb{Z}$. This is because ψ is a generalized character of G. Therefore, $f(x)$ takes on integer values, for all $x \in \mathbb{Z}$. □

If G is a cyclic group, then for any divisor k of $|G|$ we have $a(k) = \varphi(k)$, where φ is the Euler φ-function, and therefore, Proposition 2 implies that $1/|G|\Sigma_{k||G|}\varphi(k)x^{|G|/k} \in \mathbb{Z}$ for all $x \in \mathbb{Z}$. Hence, if G is a cyclic group of order p, p prime, then $1/p\,(x^p + (p-1)x) \in \mathbb{Z}$ which implies Fermat's little theorem $x^p \overset{p}{\equiv} x$.

In [11] and [12], a similar consideration led the authors to obtain Fermat's little theorem as a consequence. Let G be a subgroup of the symmetric group on a finite set Ω and let $H \leq G$. Let μ be a function defined on subgroups of G recursively by $\mu(H, H) = 1$ and $\mu(H, T) = -\Sigma_{H\leq K<T}\mu(H, K)$. If $H = 1$ is the trivial subgroup of G, then it is easy to prove that $\mu(1, T) = \mu(|T|)$ is the ordinary Möbius function of number theory. For a subgroup T of G, let $c(T)$ be the number of orbits of T on Ω. Then it is proved in [12] that, for any $a \in \mathbb{Z}$, we have $|H|/|N_G(H)|\Sigma_{T\geq H}\mu(H, T)a^{c(T)} \in \mathbb{Z}$. If we let H be the trivial subgroup of G, then the above formula becomes $1/|G|\Sigma_T\mu(1, T)a^{c(T)} \in \mathbb{Z}$. Now, assume that G is embedded in $\$_\Omega$ by Cayley representation. If G is assumed to be a cyclic group, then we have $1/|G|\Sigma_{k||G|}\mu(k)x^{|G|/k} \in \mathbb{Z}$. Now, if $G = \mathbb{Z}_p$, p prime,

then by the above formula, we obtain $1/p(a^p - a) \in \mathbb{Z}$ and consequently, we re-obtain Fermat's little theorem.

In the following we obtain a result about the non-vanishing of the symmetry classes of tensors which is a generalization of the known results.

Lemma 3. *Let G be a finite group and Ω a set of n elements $n \geq 2$. Assume that G acts faithfully on Ω and let V be an m-dimensional vector space over the complex field $\mathbb{C}$. If, for all $g \in G - \{1\}$, $|fix(g)| \leq l$, then for all $m \geq l + 2$ and for all $\chi \in I(G)$, we have $V_\chi^n(G) \neq 0$ (fix(g) denotes the set of fixed points of g upon its action on Ω).*

Proof. Suppose for all $g \in G - \{1\}$, $|\mathrm{fix}(g)| \leq l'$, where l' is the sharp upper bound, i.e., there is a $g \in G - \{1\}$ such that $|\mathrm{fix}(g)| = l'$. Then, we have $l' \leq l$ and $l' \leq n-2$. Consider the action of the group G on Γ_m^n and put $\gamma = (1, 2, \ldots, l', l' + 1, l' + 2, \ldots, l' + 2)$. If $m \geq l + 2$, then $m \geq l' + 2$, but $l' + 2 \leq n$, therefore $\gamma \in \Gamma_m^n$, and we can choose Δ such that $\gamma \in \Delta$. By our hypothesis, one easily obtains that $G_\gamma = \{1\}$, so $\Sigma_{g \in G_\gamma} \chi(g) = \chi(1) \neq 0$ and therefore, $\gamma \in \overline{\Delta}$. But by (4), we have $\mathrm{dim} V_\gamma^* = \chi(1)/|G_\gamma| \Sigma_{g \in G_\gamma} \chi(g) = \chi(1)^2 \neq 0$, therefore, $V_\gamma^* \neq 0$, and by (3), we have $V_\chi^n(G) \neq 0$. □

Since, for all non-identity $g \in G$, we have $|\mathrm{fix}(g)| \leq n - 2$, we obtain the following result (see [10]) which is a consequence of the above lemma.

Corollary 4. *Let G be a finite group and Ω a set of n elements, $n \geq 2$. Assume that G acts faithfully on Ω and let V be an m-dimensional vector space over the complex field $\mathbb{C}$. Then, for all $m \geq n$ and all $\chi \in I(G)$, we have $V_\chi^n(G) \neq 0$. In particular, for the subgroup G of $\$_n$, we have $V_\chi^n(G) \neq 0$, for all $m \geq n$.*

Now, we consider a special case. Suppose G is a group of order n and G acts on $\Omega = G$ by right multiplication, i.e., for all $g \in G$, $w \in \Omega$, $g \cdot w = gw$. This action is faithful, therefore, G is a subgroup of $\$_n$ and we say G is a subgroup of $\$_n$ by *Cayley representation*. By Lemmas 1 and 3, we re-obtain the following result.

Theorem 5. *Let G be a group of order n, that is, a subgroup of $\$_n$ by Cayley representation. If V is an m-dimensional vector space over the complex field $\mathbb{C}$, then for all $\chi \in I(G)$, we have*

$$\mathrm{dim} V_\chi^n(G) = \frac{\chi(1)}{n} \sum_{g \in G} \chi(g) m^{n/o(g)},$$

in particular, for all $m \geq 2$, $V_\chi^n(G) \neq 0$.

In closing, we give a necessary condition for the existence of the orthogonal basis of decomposable symmetrized tensors for $V_\chi^n(G)$ in the case where the finite group G of order n is embedded in $\$_n$ by Cayley representation.

Theorem 6. *Let G be a non-trivial group of order n, that is, a subgroup of $\$_n$ by Cayley representation. If V is an m-unitary space, $m \geq 2$, and $\chi \in I(G)$ such that $\chi(1)^2 > |G|/2$, then $V_\chi^n(G)$ does not have an orthogonal basis of decomposable symmetrized tensors.*

Proof. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of V. Suppose $V_\chi^n(G)$ has an orthogonal basis of decomposable symmetrized tensors. Then, by (3), for all $\gamma \in \overline{\Delta}$, the orbital subspace V_γ^* has an orthogonal basis of decomposable symmetrized tensors. Put $\gamma = (1, 2, \dots, 2) \in \Gamma_m^n$, then we have $G_\gamma = \{1\}$. Therefore, $\Sigma_{g\in G_\gamma}\chi(g) = \chi(1) \neq 0$, so we can assume that $\gamma \in \overline{\Delta}$.

Therefore, by the above discussion, V_γ^*, $\gamma = (1, 2, \dots, 2)$ has an orthogonal basis of decomposable symmetrized tensors. Since $\dim V_\gamma^* = \chi(1)/|G_\gamma|\Sigma_{g\in G_\gamma}\chi(g) = \chi(1)^2 = s$, we can assume that $\{e^*_{g_1\times\gamma}, e^*_{g_2\times\gamma}, \dots, e^*_{g_s\times\gamma}\}$ is an orthogonal basis of decomposable symmetrized tensors for V_γ^*.

Define the $n \times n$ complex matrix $A = [a_{ij}]$ as below:

$$a_{ij} = \frac{\chi(1)}{n}\chi(g_i g_j^{-1}),$$

where $G = \{g_1, \dots, g_n\}$. For all i, j, $1 \le i, j \le s$, by (5), we obtain

$$\begin{aligned}
a_{ij} &= \frac{\chi(1)}{n}\chi(g_i g_j^{-1}) \\
&= \frac{\chi(1)}{|G|}\sum_{\sigma\in G_\gamma}\chi(g_i\sigma g_j^{-1}) \\
&= \langle e^*_{g_j\cdot\gamma} \mid e^*_{g_i\cdot\gamma}\rangle \\
&= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{n}\chi(1) & \text{if } i = j, \end{cases} \\
&= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{s}{n} & \text{if } i = j. \end{cases}
\end{aligned}$$

Therefore, A has the form

$$A = \begin{bmatrix} \frac{s}{n}I_s & A_1 \\ A_2 & A_3 \end{bmatrix},$$

where A_1, A_2, and A_3 are matrices of sizes $s \times (n-s)$, $(n-s) \times s$, and $(n-s) \times (n-s)$, respectively, and I_s is the $s \times s$ identity matrix. If $A^2 = [b_{ij}]$, then by the generalized orthogonality relations for characters, we obtain

$$\begin{aligned}
b_{ij} &= \sum_{k=1}^{n} a_{ik}a_{kj} \\
&= \frac{\chi(1)^2}{n^2}\sum_{k=1}^{n}\chi(g_i g_k^{-1})\chi(g_k g_j^{-1}) \\
&= \frac{\chi(1)^2}{n^2}\sum_{g\in G}\chi(g)\chi(g^{-1}g_i g_j^{-1}) \\
&= \frac{\chi(1)^2}{n^2}\frac{n}{\chi(1)}\chi(g_i g_j^{-1}) \\
&= \frac{\chi(1)}{n}\chi(g_i g_j^{-1}) = a_{ij}.
\end{aligned}$$

This leads to $A^2 = A$ and therefore if we apply this condition for the block form of A, we obtain $A_1A_2 = (s/n - s^2/n^2)I_s$. Since $s \neq n$, A_1A_2 is an invertible matrix, and thereby we can easily obtain $s \leq n - s$ or $s \leq n/2$ or $\chi(1)^2 \leq |G|/2$ which is a contradiction. Thus, $V_\chi^n(G)$ does not have an orthogonal basis of decomposable symmetrized tensors. □

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